

On an algorithm for receiving Sudoku matrices

Krasimir Yordzhev

Faculty of Mathematics and Natural Sciences, South-West University
Ivan Mihaylov 66, Blagoevgrad, 2700, Bulgaria
E-mail: yordzhev@swu.bg

Abstract

This work examines the problem to describe an efficient algorithm for obtaining $n^2 \times n^2$ Sudoku matrices. For this purpose, we define the concepts of $n \times n$ Π_n -matrix and disjoint Π_n -matrices. The article, using the set-theoretical approach, describes an algorithm for obtaining n^2 -tuples of $n \times n$ mutually disjoint Π_n matrices. We show that in input n^2 mutually disjoint Π_n matrices, it is not difficult to receive a Sudoku matrix.

Keywords: Sudoku matrix; S-permutation matrix; Π_n -matrix; disjoint matrices; data type set

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1 Introduction and notation

Let n be a positive integer. Throughout $[n]$ denotes the set

$$[n] = \{1, 2, \dots, n\},$$

$$\mathcal{U}_n = [n] \times [n] = \{\langle a, b \rangle \mid a, b \in [n]\}$$

and \mathcal{V}_n denotes the set of all subsets of \mathcal{U}_n .

Let P_{ij} , $1 \leq i, j \leq n$, be n^2 square $n \times n$ matrices, whose entries are elements of the set $[n^2] = \{1, 2, \dots, n^2\}$. The $n^2 \times n^2$ matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

is called a *Sudoku matrix*, if every row, every column and every submatrix P_{ij} , $1 \leq i, j \leq n$ comprise a permutation of the elements of the set $[n^2]$, i.e., every integer $s \in \{1, 2, \dots, n^2\}$ is found just once in each row, column, and submatrix P_{ij} . Submatrices P_{ij} are called *blocks* of P .

This work is dedicated to the problem of finding an algorithm for getting all $n^2 \times n^2$ Sudoku matrices for an arbitrary integer $n \geq 2$. This task is solved for

$n = 2$ and $n = 3$ [3]. When $n > 3$, according to our information, this problem is still open. Finding algorithm to obtain $n^2 \times n^2$, $n \geq 4$ Sudoku matrices will lead to solving the problem of constructing Sudoku puzzle of higher order, which will increase the interest in this entertaining game. Here we not going to examine and compare different algorithms for solving any Sudoku puzzle. Here we will examine some algebraic properties of $n^2 \times n^2$ Sudoku matrices, which are the basis for obtaining various Sudoku puzzles.

A *binary* (or *boolean*, or *(0,1)-matrix*) is a matrix all of whose elements belong to the set $\mathfrak{B} = \{0, 1\}$. With \mathfrak{B}_n we will denote the set of all $n \times n$ binary matrices.

Two $n \times n$ binary matrices $A = (a_{ij}) \in \mathfrak{B}_n$ and $B = (b_{ij}) \in \mathfrak{B}_n$ will be called *disjoint* if there are not integers $i, j \in [n]$ such that $a_{ij} = b_{ij} = 1$, i.e. if $a_{ij} = 1$ then $b_{ij} = 0$ and if $b_{ij} = 1$ then $a_{ij} = 0$.

A matrix $A \in \mathfrak{B}_{n^2}$ is called an *S-permutation* if in each row, in each column, and in each block of A there is exactly one 1. Let the set of all $n^2 \times n^2$ S-permutation matrices be denoted by Σ_{n^2} .

A formula for calculating the number of all pairs of disjoint S-permutation matrices is given in [11].

S-permutation matrices and their algebraic properties have an important part in the description of the discussed in [6] algorithm.

The concept of S-permutation matrix was introduced by Geir Dahl [2] in relation to the popular Sudoku puzzle. It is well known that Sudoku matrices are special cases of Latin squares. It is widespread puzzle nowadays, which presents in the entertaining pages in most of the newspapers and magazines and in entertaining web sites. Sudoku, or Su Doku, is a Japanese word (or phrase) meaning something like Number Place.

Obviously a square $n^2 \times n^2$ matrix M with elements of $[n^2] = \{1, 2, \dots, n^2\}$ is a Sudoku matrix if and only if there are matrices $A_1, A_2, \dots, A_{n^2} \in \Sigma_{n^2}$, each two of them are disjoint and such that P can be given in the following way:

$$M = 1 \cdot A_1 + 2 \cdot A_2 + \dots + n^2 \cdot A_{n^2} \quad (1)$$

Thus, the problem to describe an efficient algorithm for obtaining all n^2 -tuples of mutually disjoint S-permutation matrices naturally arises. This work is devoted to this task. For this purpose, in the next section using the set-theoretical approach, we define the concepts of Π_n -matrix and disjoint Π_n -matrices. We will prove that so defined task can be reduced to the task of receiving all n^2 -tuples of mutually disjoint Π_n -matrices.

In section 3 we will describe an algorithm for obtaining n^2 -tuples of $n \times n$ mutually disjoint Π_n matrices and we will show that in input n^2 mutually disjoint Π_n matrices, it is not difficult to receive a Sudoku matrix. Described in this article algorithm essentially differs from the algorithm described in [3].

2 A representation of S-permutation matrices

Let n be a positive integer. If $z_1 z_2 \dots z_n$ is a permutation of the elements of the set $[n] = \{1, 2, \dots, n\}$ and let us shortly denote σ this permutation. Then in this case we will denote by $\sigma(i)$ the i -th element of this permutation, i.e. $\sigma(i) = z_i, i = 1, 2, \dots, n$.

Definition 1 Let Π_n denotes the set of all $n \times n$ matrices, constructed such that $\pi \in \Pi_n$ if and only if the following three conditions are true:

- i) the elements of π are ordered pairs of integers $\langle i, j \rangle$, where $1 \leq i, j \leq n$;
- ii) if

$$[\langle a_1, b_1 \rangle \quad \langle a_2, b_2 \rangle \quad \dots \quad \langle a_n, b_n \rangle]$$

is the i -th row of π for any $i \in [n] = \{1, 2, \dots, n\}$, then $a_1 a_2 \dots a_n$ in this order is a permutation of the elements of the set $[n]$;

- iii) if

$$\begin{bmatrix} \langle a_1, b_1 \rangle \\ \langle a_2, b_2 \rangle \\ \vdots \\ \langle a_n, b_n \rangle \end{bmatrix}$$

is the j -th column of π for any $j \in [n]$, then b_1, b_2, \dots, b_n in this order is a permutation of the elements of the set $[n]$.

The matrices of the set Π_n we will call Π_n -matrices.

From Definition 1, it follows that we can represent each row and each column of a matrix $M \in \Pi_n$ with the help of a permutation of elements of the set $[n]$.

Conversely for every $(2n)$ -tuple

$$\langle \langle \rho_1, \rho_2, \dots, \rho_n \rangle, \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle \rangle,$$

where

$$\rho_i = \rho_i(1) \rho_i(2) \dots \rho_i(n), \quad 1 \leq i \leq n$$

$$\sigma_j = \sigma_j(1) \sigma_j(2) \dots \sigma_j(n), \quad 1 \leq j \leq n$$

are $2n$ permutations of elements of $[n]$ (not necessarily different), then the matrix

$$\pi = \begin{bmatrix} \langle \rho_1(1), \sigma_1(1) \rangle & \langle \rho_1(2), \sigma_2(1) \rangle & \dots & \langle \rho_1(n), \sigma_n(1) \rangle \\ \langle \rho_2(1), \sigma_1(2) \rangle & \langle \rho_2(2), \sigma_2(2) \rangle & \dots & \langle \rho_2(n), \sigma_n(2) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \rho_n(1), \sigma_1(n) \rangle & \langle \rho_n(2), \sigma_2(n) \rangle & \dots & \langle \rho_n(n), \sigma_n(n) \rangle \end{bmatrix}$$

is matrix of Π_n . Hence

$$|\Pi_n| = (n!)^{2n} \tag{2}$$

Definition 2 We say that matrices $\pi' = [p'_{ij}]_{n \times n} \in \Pi_n$ and $\pi'' = [p''_{ij}]_{n \times n} \in \Pi_n$ are disjoint, if $p'_{ij} \neq p''_{ij}$ for every $i, j \in [n]$.

Definition 3 Let $\pi', \pi'' \in \Pi_n$, $\pi' = [p'_{ij}]_{n \times n}$, $\pi'' = [p''_{ij}]_{n \times n}$ and let the integers $i, j \in [n]$ are such that $p'_{ij} = p''_{ij}$. In this case we will say that p'_{ij} and p''_{ij} are component-wise equal elements.

Obviously two Π_n -matrices are disjoint if and only if they do not have component-wise equal elements.

Example 1 We consider the following Π_3 -matrices:

$$\begin{aligned}\pi' = [p'_{ij}] &= \begin{bmatrix} \langle 3, 1 \rangle & \langle 2, 1 \rangle & \langle 1, 2 \rangle \\ \langle 2, 3 \rangle & \langle 3, 2 \rangle & \langle 1, 1 \rangle \\ \langle 3, 2 \rangle & \langle 1, 3 \rangle & \langle 2, 3 \rangle \end{bmatrix} \\ \pi'' = [p''_{ij}] &= \begin{bmatrix} \langle 3, 2 \rangle & \langle 1, 3 \rangle & \langle 2, 1 \rangle \\ \langle 3, 3 \rangle & \langle 1, 1 \rangle & \langle 2, 2 \rangle \\ \langle 2, 1 \rangle & \langle 1, 2 \rangle & \langle 3, 3 \rangle \end{bmatrix} \\ \pi''' = [p'''_{ij}] &= \begin{bmatrix} \langle 3, 1 \rangle & \langle 1, 3 \rangle & \langle 2, 2 \rangle \\ \langle 2, 2 \rangle & \langle 3, 1 \rangle & \langle 1, 1 \rangle \\ \langle 2, 3 \rangle & \langle 1, 2 \rangle & \langle 3, 3 \rangle \end{bmatrix}\end{aligned}$$

Matrices π' and π'' are disjoint, because they do not have component-wise equal elements.

Matrices π' and π''' are not disjoint, because they have two component-wise equal elements: $p'_{11} = p'''_{11} = \langle 3, 1 \rangle$ and $p'_{23} = p'''_{23} = \langle 1, 1 \rangle$.

Matrices π'' and π''' are not disjoint, because they have three component-wise equal elements: $p''_{12} = p'''_{12} = \langle 1, 3 \rangle$, $p''_{32} = p'''_{32} = \langle 1, 2 \rangle$, and $p'_{33} = p'''_{33} = \langle 3, 3 \rangle$.

The relationship between S-permutation matrices and the matrices from the set Π_n are given by the following theorem:

Theorem 1 Let n be an integer, $n \geq 2$. Then there is one to one correspondence $\theta : \Pi_n \rightleftharpoons \Sigma_{n^2}$.

Proof. Let $\pi = [p_{ij}]_{n \times n} \in \Pi_n$, where $p_{ij} = \langle a_i, b_j \rangle$, $i, j \in [n]$, $a_i, b_j \in [n]$. Then for every $i, j \in [n]$ we construct a binary $n \times n$ matrices A_{ij} with only one 1 with coordinates (a_i, b_j) . Then we obtain the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}. \quad (3)$$

According to the properties i), ii) and iii), it is obvious that the obtained matrix A is $n^2 \times n^2$ S-permutation matrix.

Conversely, let $A \in \Sigma_{n^2}$. Then A is in the form shown in (3) and for every $i, j \in [n]$ in the block A_{ij} there is only one 1 and let this 1 has coordinates (a_i, b_j) .

For every $i, j \in [n]$ we obtain ordered pairs of integers $\langle a_i, b_j \rangle$ corresponding to these coordinates. As in every row and every column of A there is only one 1, then the matrix $\pi = [p_{ij}]_{n \times n}$, where $p_{ij} = \langle a_i, b_j \rangle$, $1 \leq i, j \leq n$, which is obtained by the ordered pairs of integers is matrix of Π_n , i.e. matrix for which the conditions i), ii) and iii) are true. \square

Corollary 1 *Let $\pi', \pi'' \in \Pi_n$ and let $A' = \theta(\pi')$, $A'' = \theta(\pi'')$, where θ is the bijection defined in Theorem 1. Then A' and A'' are disjoint if and only if π' and π'' are disjoint.*

Proof. It is easy to see that with respect of the described in Theorem 1 one to one correspondence, every pair of disjoint matrices of Π_n will correspond to a pair of disjoint matrices of Σ_{n^2} and conversely every pair of disjoint matrices of Σ_{n^2} will correspond to a pair of disjoint matrices of Π_n . \square

Corollary 2 [2] *The number of all $n^2 \times n^2$ S-permutation matrices is equal to*

$$|\Sigma_{n^2}| = (n!)^{2n} \quad (4)$$

Proof. It follows immediately from Theorem 1 and formula (2). \square

3 Description of the algorithm

Algorithm 2 *Receive n^2 mutually disjoint Π_n -matrices.*

INPUT: Integer n

OUTPUT: $P_1, P_2, \dots, P_{n^2} \in \Pi_n$ such that P_i and P_j are disjoint when $i \neq j$

1. Construct $n \times n$ arrays P_1, P_2, \dots, P_n whose entries assume values of the set \mathcal{V}_n ;
2. Initialize all entries of P_1, P_2, \dots, P_n with \mathcal{U}_n ;
3. For every $k = 1, 2, \dots, n^2$ do loop
4. For every $i = 1, 2, \dots, n$ do loop
5. For every $j = 1, 2, \dots, n$ do loop
6. Choose $\langle a, b \rangle \in P_k[i][j]$;
7. $P_k[i][j] = \{\langle a, b \rangle\}$;
8. For every $t = k + 1, k + 2, \dots, n^2$ from the set $P_t[i][j]$ remove the element $\langle a, b \rangle$;
9. For every $t = j + 1, j + 2, \dots, n$ from the set $P_t[i][j]$ remove all elements $\langle x, y \rangle$ such that $x = a$;

10. For every $t = i + 1, i + 2, \dots, n$ from the set $P_t[i][j]$ remove all elements $\langle x, y \rangle$ such that $y = b$;
 end loop 5;
 end loop 4;
 end loop 3.

Algorithm 3 Receive a S -permutation matrix from a Π_n -matrix.

INPUT: $P = [\langle a_k, b_l \rangle]_{n \times n} \in \Pi_n$, $1 \leq k, l \leq n$.

OUTPUT: $S = [s_{ij}]_{n^2 \times n^2} \in \Sigma_{n^2}$, $1 \leq i, j \leq n^2$.

1. Construct an $n^2 \times n^2$ integer array $S = [s_{ij}]$, $1 \leq i, j \leq n^2$ and initialize $s_{ij} = 0$ for all $i, j \in \{1, 2, \dots, n^2\}$;
2. For every $k, l \in \{1, 2, \dots, n\}$ do loop
3. $i = (k - 1) * n + a_k$;
4. $j = (l - 1) * n + b_l$;
5. $s_{ij} = 1$
 end loop.

Algorithm 4 Receive Sudoku matrices.

INPUT: Integer n .

OUTPUT: Sudoku matrix A .

1. Get n^2 mutually disjoint Π_n matrices P_1, P_2, \dots, P_{n^2} (Algorithm 2);
2. For every $k = 1, 2, \dots, n^2$ from P_k receive $S_k \in \Sigma_{n^2}$ (Algorithm 3);
3. $A = 1 * S_1 + 2 * S_2 + \dots + n^2 * S_{n^2}$.

4 Conclusion and remarks

- Described in section 3 algorithms will work more efficiently if the programmer uses programming languages and programming environments with integrated tools for working with data structure **set** [1, 4, 5, 7, 8, 9].
- If in item 6 of Algorithm 2 we choose ordered pair $\langle a, b \rangle \in P_k[i][j]$ randomly, then we will get a random Sudoku matrix [10]. Thus we tested the effectiveness of the algorithm.
- If in item 6 of Algorithm 2 we choose all ordered pairs $\langle a, b \rangle \in P_k[i][j]$, then finally we will get all $n^2 \times n^2$ Sudoku matrices. We do not know a general formula for finding the number θ_n of all $n^2 \times n^2$ Sudoku matrices for each integer $n \geq 2$. We consider that this is an open mathematical problem. Using a computer program based on described in section 3 algorithms, we calculated that when $n = 2$, there are $\theta_2 = 288$ number of 4×4 Sudoku

matrices. This number coincides with our results obtained using other methods described in [12]. In [3], it has been shown that there are exactly $\theta_3 = 9! \cdot 72^2 \cdot 2^7 \cdot 27\,704\,267\,971 = 6\,670\,903\,752\,021\,072\,936\,960$ number of 9×9 Sudoku matrices. The next step is to calculate the number θ_4 of 16×16 Sudoku matrices.

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